MTH 166

Lecture-4

Equs. of First Order and Higher Degree

Topic:

Equations of First Order and Higher Degree

Learning Outcomes:

1. What are the Equations of First Order and Higher Degree.

2. How to solve these equations first order and higher degree.

Equations of First Order and Higher Degree:

Let $\frac{dy}{dx} = p$ (A standard notation)

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = p^2$ $\left(Not\ \frac{d^2y}{dx^2}\right)$

$\Rightarrow \left(\frac{dy}{dx}\right)^3 = p^3$ $\left(Not\ \frac{d^3y}{dx^3}\right)$

$\Rightarrow \left(\frac{dy}{dx}\right)^4 = p^4$ and so on…

These types of equations can also be written as: $f(x, y, p) = 0$

<u>Eqations Solvable for p:</u>

<u>Find the general solution of following differential equations:</u>

<u>Problem 1.</u> $\boldsymbol{y\left(\frac{dy}{dx}\right)^2 + (x - y)\left(\frac{dy}{dx}\right) - x = 0}$ **(1)**

<u>Solution:</u> Let $\frac{dy}{dx} = p$

Equation (1) can be re-written as:

$$yp^2 + (x - y)p - x = 0$$

$$\Rightarrow yp^2 + px - py - x = 0$$

$$\Rightarrow p(yp + x) - 1(yp + x) = 0$$

$$\Rightarrow (yp + x)(p - 1) = 0$$

Here $(yp + x) = 0$

$\Rightarrow y\frac{dy}{dx} + x = 0$

$\Rightarrow ydy + xdx = 0$

$\Rightarrow \int ydy + \int xdx = c$

$\Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = c_1$

$\Rightarrow (y^2 + x^2 - 2c_1) = 0$

Also $(p - 1) = 0$

$\Rightarrow \frac{dy}{dx} - 1 = 0$

$\Rightarrow dy - dx = 0$

$\Rightarrow \int dy - \int dx = 0$

$\Rightarrow y - x = c_2$

$\Rightarrow (y - x - c_2) = 0$

So, the general solution of equation (1) is given by:

$(y^2 + x^2 - 2c_1)\,(y - x - c_2) = 0$ **Answer.**

Problem 2. $\boldsymbol{xy\left(\frac{dy}{dx}\right)^2 - \left(x^2 + y^2\right)\left(\frac{dy}{dx}\right) + xy = 0}$ **(1)**

Solution: Let $\frac{dy}{dx} = p$

Equation (1) can be re-written as:

$$xyp^2 - (x^2 + y^2)p + xy = 0$$

$$\Rightarrow xyp^2 - x^2p - y^2p + xy = 0$$

$$\Rightarrow xp(yp - x) - y(yp - x) = 0$$

$$\Rightarrow (yp - x)(xp - y) = 0$$

Here $(yp - x) = 0$

$$\Rightarrow y\frac{dy}{dx} - x = 0$$

$$\Rightarrow ydy - xdx = 0$$

$$\Rightarrow \int ydy - \int xdx = c$$

$$\Rightarrow \frac{y^2}{2} - \frac{x^2}{2} = c_1$$

$$\Rightarrow (y^2 - x^2 - 2c_1) = 0$$

Here $(xp - y) = 0$

$$\Rightarrow x\frac{dy}{dx} - y = 0$$

$$\Rightarrow xdy - ydx = 0$$

$$\Rightarrow xdy = ydx$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\Rightarrow \log y = \log x + \log c_2$$

$$\Rightarrow \log y = \log xc_2$$

$$\Rightarrow y = xc_2 \quad \Rightarrow (y - xc_2) = 0$$

So, the general solution of equation (1) is given by:

$(y^2 - x^2 - 2c_1)\,(y - xc_2) = 0$ **Answer.**

Problem 3. $\boldsymbol{p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0}$ (1)

Solution: $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$ (1)

$\Rightarrow p[p^2 + 2xp - y^2p - 2xy^2] = 0$

$\Rightarrow p[p(p + 2x) - y^2(p + 2x)] = 0$

$\Rightarrow p(p + 2x)(p - y^2) = 0$

Here $p = 0$ $\Rightarrow \frac{dy}{dx} = 0$ $\Rightarrow y = c_1$ $\Rightarrow (y - c_1) = 0$

Also $(p + 2x) = 0$

$\Rightarrow \frac{dy}{dx} + 2x = 0$

$\Rightarrow dy + 2xdx = 0$ $\Rightarrow \int dy + 2\int xdx = c_2$ $\Rightarrow y + 2\left(\frac{x^2}{2}\right) = c_2$

$\Rightarrow (y + x^2 - c_2) = 0$

Also $(p - y^2) = 0$

$$\Rightarrow \frac{dy}{dx} - y^2 = 0$$

$$\Rightarrow \frac{dy}{dx} = y^2$$

$$\Rightarrow \frac{dy}{y^2} = dx$$

$$\Rightarrow \int y^{-2}\, dy = \int dx$$

$$\Rightarrow \frac{-1}{y} = x + c_3$$

$$\Rightarrow (xy + 1 + yc_3) = 0$$

So, the general solution of equation (1) is given by:

$(y - c_1)(y + x^2 - c_2)(xy + 1 + yc_3) = 0$ **Answer.**

Problem 3. $\boldsymbol{p(p+y) = x(x+y)}$ **(1)**

Solution: $p(p+y) = x(x+y)$ (1)

$\Rightarrow\ p^2 + yp - x^2 - xy = 0$

$\Rightarrow (p^2 - x^2) + y(p - x) = 0$

$\Rightarrow (p - x)(p + x) + y(p - x) = 0$

$\Rightarrow (p - x)(p + x + y) = 0$

Here $(p - x) = 0$

$\Rightarrow \frac{dy}{dx} - x = 0$

$\Rightarrow dy - xdx = 0$ $\Rightarrow \int dy - \int xdx = c_1$ $\Rightarrow y - \left(\frac{x^2}{2}\right) = c_1$

$\Rightarrow (2y - x^2 - 2c_1) = 0$

Also $(p + x + y) = 0$

$\Rightarrow \frac{dy}{dx} + x + y = 0$

$\Rightarrow (x + y)dx + dy = 0$ (2)

Comparing it with: $Mdx + Ndy = 0$

Here $M = (x + y) \quad \Rightarrow \frac{\partial M}{\partial y} = 1$

and $N = 1 \qquad \Rightarrow \frac{\partial N}{\partial x} = 0$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, So, equation (2) in non-exact.

Here $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1-0}{1} = 1 = x^0 = f(x)$ which can be considered as a function of x.

So, $I.F. = e^{\int f(x)dx} = e^{\int 1dx} = e^x$

Multiplying equation (2) by I.F.

$[(x + y)dx + dy] \times e^x = 0$

$\Rightarrow (xe^x + ye^x)dx + e^x dy = 0$

Which is an exact differential equation.

Solution: $\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = c_2$

$\Rightarrow \int_{y=con.} (xe^x + ye^x)dx + 0 = c_2$

$\Rightarrow e^x(x - 1) + ye^x = c_2$

$\Rightarrow (e^x(x - 1) + ye^x - c_2) = 0$

So, the general solution of equation (1) is given by:

$(2y - x^2 - 2c_1)(e^x(x - 1) + ye^x - c_2) = 0$ **Answer.**